RIGOROUS RESULTS in SPACE-SPACE NONCOMMUTATIVE QUANTUM FIELD THEORY

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The axiomatic approach based on Wightman functions is developed in noncommutative quantum field theory. We have proved that the main results of the axiomatic approach remain valid if the noncommutativity affects only the spatial variables.

1. Introduction

The axiomatic approach developed in the works by Wightman, Jost, Bogoliubov, Haag, and others (see [1]–[4] and the references therein) allowed stating quantum field theory as a consistent mathematically rigorous science. This approach was first generalized to gauge field theory in the works by Morchio and Strocchi [5], [6].

In the framework of this approach, some fundamental results were obtained based on general principles of the theory, for example, the CPT theorem and the spinstatistics theorem were proved. Analytic properties of scattering amplitudes with respect to energy and transferred momentum were established, based on which rigorous bounds on the high-energy behavior of the amplitudes were derived (see [7]).

Widespread consideration is currently given to various generalizations of the standard quantum field theory, in particular, to noncommutative quantum field theory, whose foundation includes the coordinate commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant matrix. Interest in this kind of theory has a long history [8]. A new developmental stage is related, on one hand, to the construction of noncommutative geometry [9] and, on the other hand, to new arguments in favor of this generalization of the theory to ultrashort distances, i.e., to ultrahigh energies [10]. Furthermore, there is great interest in these theories because they are low-energy limits of string theories in some cases [11] (see [12], [13] for a survey of the main results in this research direction). The noncommutative theories naturally decompose into two classes, namely, those in which time commutes with the spatial variables, i.e. $\theta_{0i} = 0$, and those (this is the general case) in which $\theta_{0i} \neq 0$. If $\theta_{0i} = 0$, then the theory involves no difficulties related to unitarity and causality, which are characteristic of the general case [14]–[16], although there are also some versions of a consistent theory in the general case [17]. We note that the version with $\theta_{0i} = 0$ is precisely the low-energy limit of string theory [11].

The major part of calculations in noncommutative field theory is connected with perturbation theory. But a more general approach was developed in some works (see [18]–[29]). In particular, the investigations in [24], [25] have their origin in the ideas of algebraic field theory, and the consideration in [26]–[29] is based on introducing Wightman functions. The present report is devoted to the development of the approach in [26]–[29].

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As in [26]–[29], we consider the version of the theory in which $\theta_{0i} = 0$. In this case, without loss of generality [30], we can choose a representation that determines the algebra defined by relation (1) and for which only the condition $\theta_{12} = -\theta_{21} \neq 0$ holds. Consequently, we have two noncommutative coordinates, x_1 x_2, and two commutative ones, x_0 x_3. Wightman functions turn out to be analytic with respect to the commutative variables in a domain that is a two-dimensional analogue of the corresponding analyticity domain in the standard theory [3], [26]. This conclusion is based on the $SO\,(1,1)$ invariance of the theory with respect to the commutative variables. Besides, instead of the standard spectrality condition [1], [3], only its analogue for the momentum components corresponding to the commutative variables is used. The derivation of noncommutative analogues of the main results in the axiomatic approach such as the CPT theorem and the spinstatistics theorem is based on the analogue of local commutativity [28]

$$[\varphi\,(\hat{x}), \varphi\,(\hat{y})] = 0, \tag{2}$$

if $(x_0 - y_0)^2 - (x_3 - y_3)^2 < 0$.

For the noncommutative variables, it suffices to assume that the theory is invariant with respect to the reflection $I_s\,(2)$ for noncommutative coordinates. Therefore, only the existence of the $SO\,(1,1) \otimes I_s\,(2)$ symmetry and also of analogues of the local commutativity and spectrality conditions is important here. Moreover, some of the results are based only on the $SO\,(1,1)$ symmetry and on the spectrality condition.

In this report, we prove some classical results of axiomatic quantum field theory for a real scalar field. We prove the irreducibility of the set of field operators, the equivalence between the local commutativity and symmetry conditions for Wightman functions with respect to the permutation of arguments in the analyticity domain. We prove the CPT theorem and introduce the Borchers equivalence classes. Some similar results related to the construction of Borchers equivalence classes were obtained also in [31]. We consider the theorem on the spinstatistics relation as well.

We emphasize here that noncommutative analogues for the local commutativity and spectrality conditions are also valid in the commutative theory. It thus turns out that to reproduce the main results of the axiomatic approach in the commutative theory, it suffices to proceed from some assumptions that are weaker than the standard ones. Noncommutativity plays the role of a new physical reality making these weakenings of the local commutativity and spectrality conditions inevitable. Of course, the results obtained in this paper also remain true in other generalizations of the standard theory that preserve the validity of the above-mentioned analogues of the local commutativity and spectrality conditions.

This report has the following structure. Because the suggested approach is a natural generalization of the standard formalism, we begin with a presentation of the essence of this formalism. We then introduce the Wightman functions in noncommutative QFT.

We first show that the translation invariance in respect to x_0 coordinate only together with the assumption that the set of basic vectors contains only the positive energy vectors leads to the irreducibility of the set of field operators. This result is valid also if $\theta_{0i} \neq 0$. We then demonstrate that if the set of vectors in momentum space does not contains taxyons in respect to commutative variables(noncommutative spectral condition) then Wightman functions are analytical functions in respect to commutative variables in the domain similar to tubes. $SO\,(1,1)$ invariance gives us a possibility to enlarge this domain of analyticity (enlarged tubes). This domain contains the subset of Jost points, in which noncommutative LCC (local commutativity condition) is fulfilled. This fact leads to the possibility to use standard axiomatic methods and obtain above-mentioned results.

2. Main postulates

We first recall the essence of the Wightman approach in quantum field theory. To simplify the formulas, we state this approach as if the field $\varphi(x)$ were defined at a point. In reality, the operators are given by the expressions

$$\varphi_f \equiv \int \varphi(x)\, f(x)\, d\, x, \tag{3}$$

where $f(x)$ is a test function [1][3]. It is assumed that the vacuum vector Ψ_0 is cyclic or, in other words, that the vectors of the type

$$\varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n)\, \Psi_0 \tag{4}$$

form a basis in the space J under consideration.

We assume that the space J is endowed with an inner product $< \cdot, \cdot >$, which can be indefinite. The cyclicity condition for the vacuum means that each vector in the space J can be approximated arbitrarily accurately by vectors of type (4). We do not specify this condition because the only essential thing in what follows is that for all vectors, the inner product can be approximated arbitrarily accurately by a linear combination of Wightman functions

$$W(x_1, x_2, ..., x_n) = \langle \Psi_0, \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n)\, \Psi_0 \rangle. \tag{5}$$

It is clear that the inner product of two basis vectors is a Wightman function. Passing to the noncommutative space, we can introduce the functions

$$W(x_1, x_2, ..., x_n) = \langle \Psi_0, \varphi(x_1)\, \tilde{\star}\, \varphi(x_2)\, \tilde{\star} \ldots \varphi(x_n)\ \Psi_0 \rangle, \tag{6}$$

where Wightman functions are written in the commutative space with the corresponding changes in the product of operators.

As is known (see [12], [13]), the product of two arbitrary operators $\varphi(x)$ and $\psi(x)$ in the noncommutative theory is replaced with the $\star$-product (Moyal-type product)

$$\phi(\hat{x})\, \psi(\hat{x}) \to \phi(x) \star \psi(x), \tag{7}$$

where

$$\phi(x) \star \psi(x) = \phi(x)\, e^{\frac{i}{2}\theta_{\mu\nu} \frac{\overleftarrow{\partial}}{\partial x^{\mu}} \frac{\overrightarrow{\partial}}{\partial y^{\nu}}}\, \psi(y)|_{x=y}\ . \tag{8}$$

It is clear that there is an essential ambiguity in the choice of the corresponding generalization of the Moyal-type product for different points. It is remarkable that the possibility of extending the axiomatic approach to the noncommutative theory is independent of the concrete form of this generalization provided that it satisfies some general requirements. As in the usual case, we assume that the vectors of the type

$$\varphi(x_1)\, \tilde{\star}\, \varphi(x_2)\, \tilde{\star} \ldots \varphi(x_n)\ \Psi_0 \tag{9}$$

form a set of basis vectors in the space $J_{\tilde{\star}}$ under consideration and that Ψ_0 is a cyclic vector. The simplest way to choose $\tilde{\star}$ is to set $\tilde{\star} = 1$ for different neighboring points. Indeed, this choice was made in [26]. But we note that the usual operator product is used in [26] at coincident points as well, which, strictly speaking, corresponds to the commutative theory with the $SO(1,1) \otimes SO(2)$ invariance. An alternative natural choice of the $\tilde{\star}$-product is the direct generalization of the Moyal-type product to different points [13], i.e., $\tilde{\star} = \star$. This choice corresponds to the Wightman functions introduced in [27]. In this case, the noncommutativity is manifested not only at coincident points but also in their neighborhood. We thus automatically obtain the correct expression at coincident neighboring points.

The important point is the choice of test functions. We assume that as $\theta_{0i} = 0$, test functions in respect to commutative variables are standard ones. Precisely we assume that $f\,(x)$ can be represented in the form $f\,(x) = f_1\,(x_0, x_3) f_2\,(x_1, x_2)$, where $f_1\,(x_0, x_3)$ corresponds to tempered distributions and $f_2\,(x_1, x_2)$ gives rise one of Shilov- Gelfand spaces [32]. The exact determination of this space is unessential as only the above-mentioned decomposition of $f\,(x)$ is important. For simplicity we formally can consider a field operator as an operator defined in a point as all our reasonings are valid also for φ_f (see eq. (3)). It follows from expression (9) that in noncommutative theory the role of field operators plays the combination $\varphi\,(x)\,\tilde{\star}$. As we consider a real field, then

$$[\varphi\,(x)\,\tilde{\star}]^{+} = \varphi\,(x)\,\tilde{\star}\ .$$

It is convenient to denote $\varphi\,(x)\,\tilde{\star}$ as $\tilde{\varphi}\,(x)$. In new notations

$$W\,(x_1, x_2, ..., x_n) = \langle \Psi_0, \tilde{\varphi}\,(x_1) \cdots \tilde{\varphi}\,(x_n)\,\Psi_0 \rangle. \tag{10}$$

We once again stress that axiomatic results are based on the commutativity and spectrality conditions, not on a definite choice of the $\tilde{\star}$-product.

3. Irreducibility of the Field Operators Set

As in commutative case irreducibility of the field operators set in noncommutative theory is a consequence both of translation invariance of the theory and corresponding spectral conditions. In fact it is sufficient (the same is true also in commutative case) that a vacuum vector would be invariant under the translations along x_0 axis and all basic vectors in momentum space P_n satisfy the condition

$$P_0 \geq 0\ .$$

This condition and completeness of the set of basic vectors in momentum space leads to equality

$$\int d\,a\,e^{i\,p\,a}\,\langle \Phi, U\,(a)\,\Psi \rangle \neq 0, \quad \text{only if} \quad p_0 \geq 0\ , \tag{11}$$

Φ and Ψ are arbitrary vectors, $U\,(a)$ is a translation operator, a is a translation along x_0.

Let us remind the criterion of irreducibility. As $\tilde{\varphi}\,(x)$ is an unbounded operator, a proper definition of irreducibility would be:

Definition. The set of operators $\tilde{\varphi}\,(x_i)$ is irreducible if any bounded operator A, which commutes with all field operators

$$[A, \tilde{\varphi}\,(x)] = 0, \qquad \forall\, x \tag{12}$$

is the following:

$$A = C\mathbb{I}, \qquad C \in \mathbb{C}, \tag{13}$$

$\mathbb{I}$ is an identical operator.

To prove the irreducibility of the set of operators in question we consider the expression

$$\langle \Psi_0, A\,U\,(a)\,\varphi(\tilde{x}_1)\,\varphi(\tilde{x}_2)...\varphi(\tilde{x}_n)\,\Psi_0 \rangle.$$

Using translation invariance of Ψ_0 and condition (12), we can write the chain of equalities:

$$\langle \Psi_0, A\,U\,(a)\,\varphi(\tilde{x}_1)\,\varphi(\tilde{x}_2)...\varphi(\tilde{x}_n)\,\Psi_0 \rangle = \langle \Psi_0, \varphi\,(\tilde{x}_1 + a) \cdots \varphi\,(\tilde{x}_n + a)\,A\,\Psi_0 \rangle =$$

$$\langle \varphi\,(\tilde{x}_n) \cdots \varphi\,(\tilde{x}_1), U\,(-a)\,A\,\Psi_0 \rangle, \tag{14}$$

where a is a translation along x_0 axis. Let us remind that $\tilde{\varphi}(x_i)$ is a Hermitian operator. So

$$\langle A^+ \Psi_0, U(a)\,\varphi(\tilde{x}_1)...\varphi(\tilde{x}_n)\,\Psi_0\rangle = \langle \varphi(\tilde{x}_n)\cdots\varphi(\tilde{x}_1)\,\Psi_0, U(-a)\,A\,\Psi_0\rangle. \tag{15}$$

Let us consider

$$\int d\,a\,e^{i\,p\,a}\,\langle A^+ \Psi_0, U(a)\,\varphi(\tilde{x}_1)...\varphi(\tilde{x}_n)\,\Psi_0\rangle = \\ \int d\,a\,e^{i\,p\,a}\,\langle \varphi(\tilde{x}_n)\cdots\varphi(\tilde{x}_1), U(-a)\,A\,\Psi_0\rangle. \tag{16}$$

In accordance with eq. (11) the left part of eq. (16) is not zero only if $p \in \bar{V}_2^+$ and the right part is not zero only if $-p \in \bar{V}_2^+$. So if we take vector $\varphi(\tilde{x}_n)\cdots\varphi(\tilde{x}_1)\,\Psi_0$, which is the linear combination of all basic vectors excluding vacuum one, then

$$\int d\,a\,e^{i\,p\,a}\,\langle \varphi(\tilde{x}_n)\cdots\varphi(\tilde{x}_1), U(-a)\,A\,\Psi_0\rangle = 0 \quad \forall\, p. \tag{17}$$

Eq. (17) implies that

$$\langle \Phi, A\,\Psi_0\rangle = 0 \tag{18}$$

for any vector Φ, which is the linear combination of all basic vectors excluding vacuum one.

So

$$A\,\Psi_0 = C\,\Psi_0, \qquad C \in \mathbb{C}. \tag{19}$$

From eq. (19) it follows immediately that

$$A\,\varphi(\tilde{x}_1)...\varphi(\tilde{x}_n)\,\Psi_0 = C\,\varphi(\tilde{x}_1)...\varphi(\tilde{x}_n)\,\Psi_0. \tag{20}$$

As A is a bounded operator eq. (20) implies that

$$A\,\Phi = C\,\Phi, \qquad \forall\,\Phi \in J_{\tilde{\star}}. \tag{21}$$

As translation invariance is valid in noncommutative theory in the general case $\theta_{0i} \neq 0$, we can obtain the same result for general case as well.

4. Analytical Properties of Wightman Functions

As is known in commutative case analyticity of Wightman functions in the primitive domain of analyticity (tubes) is the consequence of spectral condition and translation invariance only. The extension of this domain (the Bargman–Hall–Wightman theorem) is realized in accordance with $SO(1,3)$ invariance of the theory. In noncommutative case we have the similar picture, but all assumptions concern commutative coordinates only. Thus in noncommutative case Wightman functions are analytical functions in respect to commutative coordinates. Let us stress that noncommutative coordinates remain real.

Taking into account translation invariance in respect to the commutative coordinates, it is convenient to write Wightman functions in the form

$$W(x_1, x_2, ..., x_n) = W(\xi_1, \ldots, \xi_{n-1}, X), \tag{22}$$

where X denotes the set of noncommutative variables and $x_i^1, x_i^2,\ i = 1, \ldots n$ and $\xi_i = \{\xi_i^0, \xi_i^3\}$, where $\xi_i^0 = x_i^0 - x_{i+1}^0, \xi_i^3 = x_i^3 - x_{i+1}^3$.

Let us formulate the spectral condition. We assume that basic vectors in momentum space are time-like vectors in respect to commutative coordinates, that is

$$P_n^0 \geq |P_n^3| \ . \tag{23}$$

This condition is convenient to rewrite as $P_n \in \bar{V}_2^+$, where $\bar{V}_2^+$ is a set of vectors, satisfying the condition $x^0 \geq |x^3|$. Let us recall that usual spectral condition is $P_n \in \bar{V}^+$, that is $P_n^0 \geq |\vec{P_n}|$. From condition (23) it follows that eq. (11) leads to the following property of Wightman functions

$$W\,(P_1, ..., P_{n-1}, X) = \frac{1}{(2\pi)^{n-1}} \int e^{i\,P_j\,\xi_j}\, W\,(\xi_1, ..., \xi_{n-1}, X)\, d\,\xi_1 ... d\,\xi_{n-1} = 0, \tag{24}$$

if $P_j \notin \bar{V}_2^+$. The proof of (24) is similar with the proof in commutative case [1], [3]. As

$$W\,(\xi_1, ..., \xi_{n-1}, X) = \frac{1}{(2\pi)^{n-1}} \int e^{-i\,P_j\,\xi_j}\, W\,(P_1, ..., P_{n-1}, X)\, d\,P_1 ... d\,P_{n-1} \ , \tag{25}$$

we obtain immediately that $W\,(\nu_1, \ldots, \nu_{n-1}, X)$ is an analytical function in the "tube" T_n^- :

$$\nu_i \in T_n^-, \quad \text{if} \quad \nu_i = \xi_i - i\,\eta_i,\ \eta_i \in V_2^+,\ \eta_i = \{\eta_i^0, \eta_i^3\} \ . \tag{26}$$

Let us stress that noncommutative coordinates are always real.

In accordance with the Bargmann-Hall-Wightman theorem $W\,(\nu_1, \ldots, \nu_{n-1}, X)$ is the analytical function in the domain T_n

$$T_n = \cup_{\Lambda_c}\, \Lambda_c\, T_n^- , \tag{27}$$

where $\Lambda_c \in S_c\,O\,(1,1)$ is a two-dimensional analog of complex Lorentz group. As it was mentioned above T_n contains the real points of analyticity, satisfying the condition (2) which implies that

$$W\,(\tilde{x}_1, \ldots, \tilde{x}_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) = W\,(\tilde{x}_1, \ldots, \tilde{x}_{i+1}, \tilde{x}_i, \ldots, \tilde{x}_n), \tag{28}$$

where $\tilde{x}_i$ are noncommutative Jost points , that is they satisfy the conditions

$$\left(\tilde{x}_i^0 - \tilde{x}_j^0\right)^2 - \left(\tilde{x}_i^3 - \tilde{x}_j^3\right)^2 < 0, \quad \forall\, i, j \ . \tag{29}$$

5. Relation between the Symmetry of Wightman Functions and the Local Commutativity Condition

As a consequence of local commutativity condition (29), we have

$$W\,(\tilde{x}_1, ...\tilde{x}_n) = W\,\Pi\,(\tilde{x}_1, ...\tilde{x}_n), \tag{30}$$

where $\Pi\,(\tilde{x}_1, ...\tilde{x}_n)$ is an arbitrary permutation of the points $\tilde{x}_i$. Condition (30) means that the function $W\,(z_1, ..., z_n),\ z_i = x_i + i\,y_i$, is symmetric with respect to its arguments in the domain into which T_n passes under the permutation of the points $\tilde{x}_i$ with subsequent application of transformations belonging to $S_c\,O\,(1,1)$. Hence, the local commutativity condition permits extending the analyticity domain for Wightman functions further.

We show that, conversely, the local commutativity condition follows from the symmetry of Wightman functions in the analyticity domain. For this, it suffices to prove that

$$W\,(z_1, ...z_{i-1}, \tilde{x}_i, \tilde{x}_{i+1}, z_{i+2}, ..., z_n) = W\,(z_1, ...z_{i-1}, \tilde{x}_{i+1}, \tilde{x}_i, z_{i+2}, ..., z_n), \tag{31}$$

where $\tilde{x}_i$ and $\tilde{x}_{i+1}$ are the Jost points and $z_j \in T_n^-$. Indeed, we can pass to the limit $Im\, z_j = 0$ in (31). To prove (31), we note that the symmetry of Wightman functions in T_n^- implies the relation

$$W\,(z_1', ...z_i', z_{i+1}', ..., z_n') = W\,(z_1', ...z_{i+1}', z_i', ..., z_n'). \tag{32}$$

The desired relation is proved if there is a tuple of points z_j' and a transformation $\Lambda\ \in S_c\, O\,(1,1)$ under which the points z_j' pass into $z_j, j \neq i, i+1$, and the points z_i', z_{i+1}' pass into $\tilde{x}_i\ \ \tilde{x}_{i+1}$. The corresponding transformation is similar to the one existing in the commutative theory (see Chap. 5 in [3]).

Therefore, the local commutativity condition, as in the commutative case, is equivalent to the symmetry of Wightman functions in the analyticity domain. This equivalence implies the impossibility of a simple generalization of the local commutativity condition, namely, the following assertion holds:

If

$$[\tilde{\varphi}\,(x), \tilde{\varphi}\,(y)] = 0, \quad \text{for} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < -l^2\ , \tag{33}$$

and if the usual axioms are given, then the local commutativity condition is satisfied.

Indeed, the proof of the analyticity of Wightman functions does not use the local commutativity condition, and we have the same set of Jost points in this case as before. After the consideration of the Jost points satisfying condition (33), we use the above-mentioned local commutativity condition to prove the symmetry of Wightman functions under the permutation of arguments. It remains to use the fact that the local commutativity is a consequence of the symmetry of Wightman functions. We note that the similar result was also proved in the commutative case under some more general assumptions [33].

6. CPT Theorem and Borchers Equivalence Classes

By definition, the CPT-transform operator for real field is given by the formula

$$\Theta\, \tilde{\varphi}\, \Theta^{-1} = \tilde{\varphi}\,(-x). \tag{34}$$

According to the Wigner theorem (see [1]), the constructed theory is CPT-invariant if Θ is antiunitary, i.e., if

$$\langle \Theta\, \Phi, \Theta\, \Psi \rangle = \langle \Psi, \Phi \rangle \quad \forall\, \Psi \text{ and } \Phi.$$

Because all inner products of basis vectors (9) reduce to Wightman functions, it suffices to find a condition under which

$$\langle \Theta\, \Psi_0, \Theta\, \tilde{\varphi}\,(x_1) \cdots \tilde{\varphi}\,(x_n)\, \Psi_0 \rangle = \overline{\langle \Psi_0, \tilde{\varphi}\,(x_1) \cdots \tilde{\varphi}\,(x_n)\, \Psi_0 \rangle}, \tag{35}$$

where the overbar denotes complex conjugation. We can easily show that (35) holds if

$$\langle \Psi_0, \tilde{\varphi}\,(x_1) \cdots \tilde{\varphi}\,(x_n)\, \Psi_0 \rangle = \langle \Psi_0, \tilde{\varphi}\,(-x_n) \cdots \tilde{\varphi}\,(-x_1)\, \Psi_0 \rangle. \tag{36}$$

According to the CPT theorem, relation (36) is equivalent to the weak local commutativity condition

$$\langle \Psi_0, \tilde{\varphi}\,(\tilde{x}_1) ... \tilde{\varphi}\,(\tilde{x}_n)\, \Psi_0 \rangle = \langle \Psi_0, \tilde{\varphi}\,(\tilde{x}_n) ... \tilde{\varphi}\,(\tilde{x}_1)\, \Psi_0 \rangle, \quad \tilde{x}_i \sim \tilde{x}_j\ . \tag{37}$$

Condition (37) is obviously satisfied if the local commutativity condition holds. But we stress that there are some well-known examples of fields [3] that satisfy the weak local commutativity condition but do not satisfy the usual local commutativity condition. Because the equivalence of (36) and (37) has been proved [26, 27, 28], we do not dwell on it. We only note that in this case, the $I_s\,(2)$

invariance of the theory with respect to the noncommutative coordinates is used essentially. In the usual situation, the invariance of Wightman functions under complex Lorentz transformations includes the invariance under the change of sign for all coordinates. In the noncommutative case, the $S_c O(1,1)$ invariance means the related invariance only under the change of sign for the commutative coordinates. The invariance of Wightman functions under the change of sign of noncommutative coordinates follows from the $I_s(2)$ invariance.

Let us point out that CPT theorem is valid in any $SO(1,1) \otimes I_s(2)$ invariant theory if only Wightman functions are analytical ones in the above-mentioned domain.

We now establish conditions under which two operators $\tilde{\varphi}(x)$ and $\tilde{\psi}(x) \equiv \psi(x)\tilde{\star}$ have a common CPTtransform operator. We show that in this situation as well as in the commutative case, the above-mentioned property holds under the condition that

$$\langle\Psi_0, \tilde{\varphi}(\tilde{x}_1)\ldots\tilde{\varphi}(\tilde{x}_i)\,\tilde{\psi}(x)\,\tilde{\varphi}(\tilde{x}_{i+1})\ldots,\tilde{\varphi}(\tilde{x}_n)\,\Psi_0\rangle = \langle\Psi_0, \tilde{\varphi}(\tilde{x}_n)\ldots\tilde{\varphi}(\tilde{x}_{i+1})\,\tilde{\psi}(x)\,\tilde{\varphi}(\tilde{x}_i)\ldots\tilde{\varphi}(\tilde{x}_1)\,\Psi_0\rangle, \tag{38}$$

if $\tilde{x}_i \sim \tilde{x}_j,\ x \sim \tilde{x}_j\ \forall\, i,j$. We assume here that the field $\tilde{\psi}(x)$ satisfies the conditions under which the domain of analyticity of the functions in (38) is the same as that for the Wightman function of the field $\tilde{\varphi}(x)$, but Ψ_0 need not be a cyclic vector for the fields $\tilde{\psi}(x)$. Moreover, we assume that the $I_s(2)$ invariance with respect to the noncommutative variables holds.

In this case, by analogy with the implication under which (36) follows from (37), the relation

$$\langle\Psi_0, \tilde{\varphi}(x_1)\ldots\tilde{\varphi}(x_i)\,\tilde{\psi}(x)\,\tilde{\varphi}(x_{i+1})\ldots,\tilde{\varphi}(x_n)\,\Psi_0\rangle = \langle\Psi_0, \tilde{\varphi}(-x_n)\ldots\tilde{\varphi}(-x_{i+1})\,\tilde{\psi}(-x)\,\tilde{\varphi}(-x_i)\ldots\tilde{\varphi}(-x_1)\,\Psi_0\rangle \tag{39}$$

is implied by condition (38) for arbitrary $x, x_1, \ldots x_n$. It can be easily seen that (39) can be written in the form

$$\langle\Phi, \tilde{\psi}(x)\,\Psi\rangle = \langle\Theta\,\Psi, \tilde{\psi}(-x)\,\Theta\,\Phi\rangle, \tag{40}$$

where $\Phi = \tilde{\varphi}(x_i)\ldots\tilde{\varphi}(x_1)\,\Psi_0,\ \Psi = \tilde{\varphi}(x_{i+1})\ldots\tilde{\varphi}(x_n)\,\Psi_0$.

It is easy see that by the antiunitarity of the operator Θ, we always have

$$\langle\Phi, \tilde{\psi}(x)\,\Psi\rangle = \langle\Theta\,\Psi, \Theta\,\tilde{\psi}(x)\,\Theta^{-1}\,\Theta\,\Phi\rangle. \tag{41}$$

Comparing (40) and (41) gives the desired result because Φ and Ψ are arbitrary basis vectors. In turn, condition (39), as in the case of the CPT theorem, leads to relation (38) . Furthermore, because Θ is antiunitary, $\tilde{\psi}(x)$ is a weakly local field, i.e., a field satisfying the weak local commutativity condition. Moreover, the condition of mutual weak local commutativity holds here. This means that the Wightman function containing arbitrarily many fields $\tilde{\varphi}(x_i)$ and $\tilde{\psi}(y_j)$ does not change at the Jost points under the replacement of the direct order of the arguments by the reverse order.

The most important conclusion is that the mutual weak locality condition is transitive, i.e., if each of the fields $\tilde{\psi}_1(x)$ and $\tilde{\psi}_2(x)$ is weakly local in relation to a field $\tilde{\varphi}(x)$, then they are mutually weakly local as well. Indeed, according to the previous argument, their CPT-transform operators coincide.

Hence, weakly local fields, as in the commutative case, form a Borchers equivalence class that includes the field $\tilde{\varphi}(x)$ and also all fields each of which is weakly local in relation to $\tilde{\varphi}(x)$, and these fields turn out to be weakly local between themselves in this case. The most important property of these fields is that the S-matrices for the fields belonging to a common Borchers equivalence class coincide if the limits of the fields as $x_0 \to -\infty$ or $x_0 \to +\infty$ coincide. Indeed, as is known, the S-matrix is related to the CPT-transform operators by [3]

$$S = \Theta^{-1}\,\Theta_{out} = \Theta_{in}^{-1}\,\Theta, \tag{42}$$

where Θ_{in} and Θ_{out} are the corresponding operators for the asymptotic fields.

It can be shown that the local commutativity condition is also transitive. This means that two fields each of which is local in relation to a field $\tilde{\varphi}\,(x)$ for which Ψ_0 is a cyclic vector are mutually local. The proof of this property is similar to the proof in the commutative case [3].

7. Spin-statistics Theorem

The theorem on the spinstatistics relation for a complex scalar field can be reduced to the corresponding theorem for a Hermitian field using the following lemma.

Lemma

If the commutation relations

$$[\tilde{\varphi}\,(x), \tilde{\psi}\,(y)] = 0, \qquad x \sim y, \tag{43}$$

$$\{\tilde{\varphi}\,(x), \tilde{\psi}^{+}\,(y)\} = 0, \qquad x \sim y. \tag{44}$$

hold and if $\tilde{\varphi}\,(x) \neq 0$, then $\tilde{\psi}\,(y) \equiv 0$.

As in the usual case, we assume that the operators $\tilde{\varphi}\,(x)$ and $\tilde{\psi}\,(y)$ act in a subspace with a positive definite metric.

The proof of the Lemma essentially uses the cluster properties of Wightman functions. In view of local commutativity condition (28), the relations

$$\begin{aligned} &\langle \Psi_0, \tilde{\varphi}\,(x_1)\ldots\tilde{\varphi}\,(x_i)\,\tilde{\varphi}\,(x_{i+1}+\lambda\,a)\ldots,\tilde{\varphi}\,(x_n+\lambda\,a)\,\Psi_0\rangle \to \\ &\qquad \langle \Psi_0, \tilde{\varphi}\,(x_1)\ldots\tilde{\varphi}\,(x_i)\,\Psi_0\rangle\langle \Psi_0, \tilde{\varphi}\,(x_{i+1})\ldots\tilde{\varphi}\,(x_n)\,\Psi_0\rangle, \end{aligned} \tag{45}$$

are sure to hold if $\lambda \to \infty$, $a_0^2 - a_3^2 = -1$. The proof of (45) reduces to the corresponding proof in the commutative case [1].

To prove the lemma, it suffices to note that according to (43) and (44), we have

$$\begin{aligned} &\langle \Psi_0, \tilde{\varphi}^{+}\,(x)\,\tilde{\psi}^{+}\,(y)\,\tilde{\psi}\,(y)\,\tilde{\varphi}\,(x)\,\Psi_0\rangle = -\,\langle \Psi_0, \tilde{\varphi}^{+}\,(x)\,\tilde{\varphi}\,(x)\,\tilde{\psi}^{+}\,(y)\,\tilde{\psi}\,(y)\,\Psi_0\rangle \to \\ &\qquad -\,\langle \Psi_0, \tilde{\varphi}^{+}\,(x)\,\tilde{\varphi}\,(x)\,\Psi_0\rangle\langle \Psi_0, \tilde{\psi}^{+}\,(y)\,\tilde{\psi}\,(y)\,\Psi_0\rangle, \end{aligned} \tag{46}$$

for $x \sim y$ if $y = x + \lambda\,a$, $\lambda \to \infty$, $a_0^2 - a_3^2 = -1$. Noting that the original Wightman function is positive and that the third expression in (46) is negative, we conclude that one of the relations

$$\tilde{\varphi}\,(x)\,\Psi_0 = 0 \quad \text{or} \quad \tilde{\psi}\,(y)\,\Psi_0 = 0. \tag{47}$$

holds. We show that the first relation in (47) means that $\tilde{\varphi}\,(x) \equiv 0$ and the other implies that $\tilde{\psi}\,(y) \equiv 0$. For simplicity we consider the proof for real field $\tilde{\varphi}\,(x)$, our arguments can be easily extended to a complex field. Indeed, for instance, let the second relation in (47) hold. In view of (43) and (44), we then have

$$\langle \Psi_0, \tilde{\varphi}\,(\tilde{x}_1)\ldots\tilde{\varphi}\,(\tilde{x}_i)\,\tilde{\psi}\,(y)\,\tilde{\varphi}\,(\tilde{x}_{i+1})\ldots,\tilde{\varphi}\,(\tilde{x}_n)\,\Psi_0\rangle = 0, \tag{48}$$

if $y \sim \tilde{x}_j$, $\forall\ j$. Because vanishing of the Wightman function at the Jost points means that it is identically zero, we have

$$\langle \Psi_0, \tilde{\varphi}\,(x_1)\ldots\tilde{\varphi}\,(x_i)\,\tilde{\psi}\,(y)\,\tilde{\varphi}\,(x_{i+1})\ldots,\tilde{\varphi}\,(x_n)\,\Psi_0\rangle = 0, \quad \forall\ x_i \tag{49}$$

for all x_i. Because Ψ_0 is a cyclic vector for the field $\tilde{\varphi}\,(x)$, we conclude that $\tilde{\psi}\,(y) \equiv 0$. The same argument also applies to $\tilde{\varphi}\,(x)$. By assumption, $\tilde{\varphi}\,(x) \neq 0$ and the assertion of the lemma is thus proved.

We next apply the lemma to the case $\tilde{\psi}\,(x) = \tilde{\varphi}\,(x)$ and prove that if

$$\{\tilde{\varphi}\,(x), \tilde{\varphi}^{+}\,(y)\} = 0, \qquad x \sim y, \tag{50}$$

then $\tilde{\varphi}\,(x) \equiv 0$ then $\tilde{\varphi}\,(x) \equiv 0$. For this, we consider the functions

$$W_1\,(x-y) = \langle \Psi_0, \tilde{\varphi}\,(x)\,\tilde{\varphi}^{+}\,(y)\,\Psi_0 \rangle,$$

$$W_2\,(x-y) = \langle \Psi_0, \tilde{\varphi}^{+}\,(x)\,\tilde{\varphi}\,(y)\,\Psi_0 \rangle.$$

Using the argument presented in [27] for these functions, we conclude that (50) implies that $\tilde{\varphi}\,(x)\,\Psi_0 = 0$, i.e. $\tilde{\varphi}\,(x) \equiv 0$.

8. Conclusion

The presented investigation shows that whereas there are analogues of local commutativity (28) and spectrality (24), the $SO\,(1,1) \otimes I_s\,(2)$ invariance of the theory suffices for deriving the main results in the axiomatic approach. A physically interesting version of the latter theory is the noncommutative quantum field theory in which time commutes with the spatial coordinates.

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